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CONSTRUCTION OF REFINED APPLIED THEORIES FOR A SHELL OF ARBITRARY SHAPE*

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Basic equations of the theory of elasticity are given in a semi-orthogonal curvilinear coordinate system in which one of the families of the coordinate surfaces is parallel to the middle surface of the shell.

Symbolic notation of Lur'e /1/ is used to obtain a solution of the problem of the theory of elasticity for a shell, in terms of a series in powers of the normal coordinate. The solution is then used to reduce the three-dimensional problem to two dimensions and to express all characteristic features of the stress and deformation states of the shell in terms of six functions, namely the coordinates of the displacement and stress vectors defined on the middle surface. Use of the first two terms of the series obtained yields an applied theory free of any hypotheses and intended for removing the external load from the front surface of the shell. A similar approach to the problem of constructing applied theories was first used in /2—4/ which made wide use of the resources of tensor analysis.

1. Semiorthogonal curvilinear coordinate system. We introduce the following notation. V is the region of space occupied by the material of the shell, $\mathbf{R}$ is radius vector of the running point (x, y, z) of this region, S is the middle surface of the shell, $\mathbf{r} = \mathbf{r}\ (\alpha, \beta)$ is some orthogonal parametrization of this surface and $\mathbf{n}$ is the unit vector normal to the surface S. Within the notation used, the equation

$$\mathbf{R} = \mathbf{r} + \mathbf{n}t \tag{1.1}$$

determines, in the region V, a curvilinear α, β, t-coordinate system. By virtue of (1.1) the arc length element ds is given by the quadratic differential form

$$ds^2 = d\mathbf{R}^2 = (d\mathbf{r} + td\mathbf{n} + \mathbf{n}dt)^2 \tag{1.2}$$

Let us consider three quadratic forms associated with the surface S: $d\mathbf{r}^2$, $-d\mathbf{r}d\mathbf{n}$ and $d\mathbf{n}^2$. We have

$$d\mathbf{r}^2 = Ed\alpha^2 + Gd\beta^2,\quad d\mathbf{n}^2 = -Hd\mathbf{r}d\mathbf{n} - k_1k_2d\mathbf{r}^2,\quad -d\mathbf{r}d\mathbf{n} = Ld\alpha^2 + 2Md\alpha d\beta + Nd\beta^2\ \ (H = k_1 + k_2) \tag{1.3}$$

Here k_1 and k_2 are the principal curvatures of the surface S, and E, G, L, M and N denotes the coefficients of the first and second quadratic form. Now, substituting (1.3) into the right-hand side of (1.2) and taking into account the fact that

$$m^2 = k_\alpha k_\beta - k_1k_2,\quad H = k_\alpha + k_\beta,\quad k_\alpha = L / E,\quad m = M / \sqrt{EG},\quad k_\beta = N / G \tag{1.4}$$

we obtain

$$ds^2 = g_{ik}\,dx^idx^k\quad (\alpha \equiv x^1,\ \beta \equiv x^2,\ t \equiv x^3),\quad g_{11} = E\,[(1 - k_\alpha t)^2 + m^2t^2],\quad g_{22} = G\,[(1 - k_\beta t)^2 + m^2t^2] \tag{1.5}$$

$$g_{33} = 1,\quad g_{12} = m\sqrt{EG}\,(Ht^2 - 2t),\quad g_{13} = g_{23} = 0,\quad g = \det\|g_{ik}\| = EG\,(1 - k_1t)^2(1 - k_2t)^2$$

Here the quantities k_α and m (k_β and $-m$) are, respectively, the normal curvature and the geodesic torsion of the surface S in the direction of the coordinate line $\beta = \mathrm{const}$ ($\alpha = \mathrm{const}$).

If the coordinate grid (α, β) on S coincides with the lines of curvature, then the function $g_{12} = 0$ and the α, β, t- coordinate system will be orthogonal. In this connection we note the class of shells the middle surface of which is formed by the motion of a plane line l_1 (generatrix) along the spatial line l_2 (directrix).

Let $x = x(u)$, $y = y(u)$ and $\mathbf{r}_0 = \mathbf{r}_0(v)$ represent a natural parametrization of the lines l_1 and l_2 respectively; $\mathbf{n}_0 = \mathbf{n}_0(v)$ and $\mathbf{b}_0 = \mathbf{b}_0(v)$ the unit vectors of the principal normal and binormal of the line l_2, and $k = k(v)$, $\tau = \tau(v)$ its curvature and torsion. Then the parametrization

$$\mathbf{r}(u, v) = \mathbf{r}_0 + \mathbf{n}_0m_1 + \mathbf{b}_0m_2,\quad m_1 = x\cos\psi + y\sin\psi,\quad m_2 = y\cos\psi - x\sin\psi,\quad \psi = \int\tau(v)\,dv$$

specifies, on the surface S, a coordinate grid (u, v) consisting of the lines of curvature, with the lines $v = \mathrm{const}$ being geodesic. In the u, v, t-coordinate system the quadratic form (1.5) has the form

$$ds^2 = (1 - k_1t)^2\,du^2 + G\,(1 - k_2t)^2\,dv^2 + dt^2,\quad G = (1 - km_1)^2,\quad k_1 = x'' / y',\quad k_2 = (m_2)_u'\,k / (1 - km_1)$$

2. Basic equations of the theory of elasticity in curvilinear coordinates Let us consider the second fundamental problem of the theory of elasticity when the values

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of the vector of intensity of external forces $\mathbf{q}$ are given on the surface S^* enclosing the shell. The displacement vector $\mathbf{U}$ satisfies in the region V the equilibrium equation

$$(\varkappa - 1)\,\mathrm{rotrot}\,\mathbf{U} + \mathrm{grad}\,\mathrm{div}\,\mathbf{U} = 0 \quad (\varkappa = 1/(2-2\nu)) \tag{2.1}$$

with the boundary condition S^* given.

Let us further consider the orthonormal local basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ where $\mathbf{i}_2$ and $\mathbf{i}_3$ are unit tangents to the coordinate lines x^2, x^3, and $\mathbf{i}_1 = \mathbf{i}_2 \times \mathbf{i}_3$ is the unit normal to the coordinate surface $x^1 = \mathrm{const}$. The unit vectors $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ satisfy the relations

$$\begin{gathered}
\mathbf{e}^1 = \mathbf{i}_1\sqrt{g_{22}/g}, \quad \mathbf{e}^2 = -\mathbf{i}_1 g_{12}/\sqrt{g g_{22}} + \mathbf{i}_2/\sqrt{g_{22}}, \quad \mathbf{e}^3 = \mathbf{i}_3 \\
D^1\mathbf{i}_1 = \mathbf{i}_3 z_3 - \mathbf{i}_2 z_2, \quad D^2\mathbf{i}_1 = \mathbf{i}_2 z_1 + \mathbf{i}_3 z_0, \quad D^3\mathbf{i}_1 = \mathbf{i}_2 z_0 \\
D^1\mathbf{i}_2 = \mathbf{i}_1 z_2 + \mathbf{i}_3 z_0, \quad D^2\mathbf{i}_2 = \mathbf{i}_3 z_4 - \mathbf{i}_1 z_1, \quad D^3\mathbf{i}_2 = -\mathbf{i}_1 z_0 \\
D^1\mathbf{i}_3 = -\mathbf{i}_1 z_3 - \mathbf{i}_2 z_0, \quad D^2\mathbf{i}_3 = -\mathbf{i}_1 z_0 - \mathbf{i}_2 z_4, \quad D^3\mathbf{i}_3 = 0, \quad \nabla \equiv \mathbf{e}^i\partial/\partial x^i \equiv \mathbf{i}_k D^k \\
D^1 = \sqrt{g_{22}/g}\,\partial/\partial\alpha - (g_{12}/\sqrt{g g_{22}})\,\partial/\partial\beta, \; D^2 = (\sqrt{g_{22}})^{-1}\,\partial/\partial\beta, \quad D^3 = \partial/\partial t \\
z_0 = mG/g_{22}, \quad z_1 = (\sqrt{g})^{-1}[(\sqrt{g_{22}})_\alpha' - (g_{12}/\sqrt{g_{22}})_\beta'], \quad z_4 = \\
-(\ln\sqrt{g_{22}})_t', \; z_2 = (\sqrt{g_{22}})^{-1}(\ln\sqrt{g/g_{22}})_\beta', \quad z_3 = g_1 + g_2 - z_4, \quad g_i = k_i/(1-k_i t)
\end{gathered} \tag{2.2}$$

The functions z_i conform to the Peterson—Codacci equations

$$\begin{gathered}
D^2 z_3 - D^1 z_0 = 2z_0 z_1 + z_2(z_4 - z_3), \quad (z_1)_t' = z_1 z_4 + D^2 z_0 \\
D^1 z_4 - D^2 z_0 = 2z_0 z_2 + z_1(z_3 - z_4), \quad (z_4)_t' = z_4^2 - z_0^2 \\
g_1 g_2 = -z_1^2 - z_2^2 - D^1 z_1 - D^2 z_2, \quad (z_0)_t' = 2z_0 z_4
\end{gathered} \tag{2.3}$$

Let $\mathbf{p}_k$ denote the stress vector at the surface with normal $\mathbf{i}_k$. Then the physical components of the stress tensor σ_{sk}^* can be found using the representation of the stress vector $\mathbf{p}_k$ in terms of the displacement vector $\mathbf{U}$. We have

$$\mathbf{p}_k = \mu\left(2D^k\mathbf{U} + \mathbf{i}_k \times \mathrm{rot}\,\mathbf{U} + \frac{2\nu}{1-2\nu}\mathbf{i}_k\,\mathrm{div}\,\mathbf{U}\right), \quad \left(\mathbf{p}_k = \sum_{s=1}^{3}\sigma_{sk}^*\mathbf{i}_s, \quad \mathbf{U} = \sum_{s=1}^{3} u_s^*\mathbf{i}_s\right) \tag{2.4}$$

Here ν in the Poisson's ratio and μ is the shear modulus. Now, taking into account the relations (2.2), we obtain

$$\begin{gathered}
D^1\mathbf{U} = \mathbf{i}_1(D^1u_1^* + u_2^*z_2 - u_3^*z_3) + \ldots \\
D^2\mathbf{U} = \mathbf{i}_1(D^2u_1^* - u_2^*z_1 - u_3^*z_0) + \mathbf{i}_2(D^2u_2^* + u_1^*z_1 - u_3^*z_4) + \ldots \\
D^3\mathbf{U} = \mathbf{i}_1[(u_1^*)_t' - u_2^*z_0] + \mathbf{i}_2[(u_2^*)_t' + u_1^*z_0] + \mathbf{i}_3(u_3^*)_t' \\
\mathrm{rot}\,\mathbf{U} = \nabla\times\mathbf{U} = \sum_{k=1}^{3}\omega_k\mathbf{i}_k, \quad \omega_1 = D^2u_3^* - (u_2^*)_t' + u_2^*z_4 \\
\omega_2 = (u_1^*)_t' - D^1u_3^* - 2u_2^*z_0 - u_1^*z_3, \quad \omega_3 = \theta(u_2^*, -u_1^*)
\end{gathered} \tag{2.5}$$

$$\mathrm{div}\,\mathbf{U} = \nabla\cdot\mathbf{U} = \theta(u_1^*, u_2^*) + (u_3^*)_t' - u_3^*(z_3 + z_4), \quad \theta(w_1, w_2) \equiv (D^1 + z_1)w_1 + (D^2 + z_2)w_2 \tag{2.6}$$

Substituting into (2.4) the corresponding quantities from (2.5) and (2.6) we obtain, for $\mathbf{i}_k = \mathbf{i}_3$,

$$\begin{gathered}
\sigma_{31} = (u_1^*)_t' + u_1^*z_3 + D^1u_3^*, \quad \sigma_{32} = (u_2^*)_t' + u_2^*z_4 + 2u_1^*z_0 + D^2u_3^* \\
\sigma_{33} = \frac{2-2\nu}{1-2\nu}(u_3^*)_t' + \frac{2\nu}{1-2\nu}[\theta(u_1^*, u_2^*) - u_3^*(z_3 + z_4)]
\end{gathered} \tag{2.7}$$

where $\sigma_{sk} = \sigma_{sk}^*/\mu$ are dimensionless stresses. Let us solve the equations (2.7) for the derivatives $(u_k^*)_t'$. We have

$$\begin{gathered}
(u_1^*)_t' = \sigma_{31} - u_1^*z_3 - D^1u_3^*, \qquad (u_2^*)_t' = \sigma_{32} - u_2^*z_4 - 2u_1^*z_0 - D^2u_3^* \\
(u_3^*)_t' = (1-\varkappa)\sigma_{33} + (1-2\varkappa)[\theta(u_1^*, u_2^*) - u_3^*(z_3 + z_4)]
\end{gathered} \tag{2.8}$$

Replacing in (2.6) the derivative $(u_3^*)_t'$ by its expression from (2.8), we obtain

$$\mathrm{div}\,\mathbf{U} = (1-\varkappa)\theta^*, \quad \theta^* = \sigma_{33} + 2[\theta(u_1^*, u_2^*) - u_3^*(z_3 + z_4)] \tag{2.9}$$

when $\mathbf{i}_k = \mathbf{i}_2, \mathbf{i}_1$ we obtain, from (2.4) with (2.5) and (2.9) taken into account,

$$\begin{gathered}
\sigma_{21} = (D^1 - z_1)u_2^* + (D^2 - z_2)u_1^* - 2u_3^*z_0 \\
\sigma_{22} = 2D^2u_2^* + 2u_1^*z_1 - 2u_3^*z_4 + (2\varkappa - 1)\theta^* \\
\sigma_{11} = 2D^1u_1^* + 2u_2^*z_2 - 2u_3^*z_3 + (2\varkappa - 1)\theta^*
\end{gathered} \tag{2.10}$$

To find rotrot $\mathbf{U}$, we replace the coordinates u_k^* in (2.5) by the corresponding coordinates ω_k. We have

$$\operatorname{rot}\operatorname{rot}\mathbf{U}=\sum_{k=1}^{3}W_k\mathbf{i}_k,\quad W_1=D^2\omega_3-(\omega_2)_t'+\omega_2 z_4,\quad W_2=(\omega_1)_t'-D^1\omega_3-2\omega_2 z_0-\omega_1 z_3,\quad W_3=\theta(\omega_2,\ -\omega_1) \tag{2.11}$$

Substituting (2.11) into (2.1), we arrive at the following equations of equilibrium in terms of the displacements:

$$(\varkappa-1)\,W_k+D^k\operatorname{div}\mathbf{U}=0\qquad(k=1,\ 2,\ 3) \tag{2.12}$$

Finally, replacing in (2.12) the derivatives $(u_k{}^*)_t'$ by the corresponding expressions from (2.8) and taking into account the relations (2.3), we obtain

$$(\sigma_{31})_t'=(z_4+2z_3)\,\sigma_{31}+2\sigma_{32}z_0-2g_1g_2u_1{}^*+D^2\theta\,(u_2{}^*,\ -u_1{}^*)+D^1(\sigma_{33}-2\varkappa\theta^*)+2\,(z_0D^2-z_4D^1)\,u_3{}^* \tag{2.13}$$

$$(\sigma_{32})_t'=(z_3+2z_4)\,\sigma_{32}-2g_1g_2u_2{}^*-D^1\theta\,(u_2{}^*,\ -u_1{}^*)+D^2(\sigma_{33}-2\varkappa\theta^*)+2\,(z_0D^1-z_3D^2)\,u_3{}^*$$

$$(\sigma_{33})_t'=2\,(z_3+z_4)(\sigma_{33}-\varkappa\theta^*)-\theta\,(\sigma_{31},\ \sigma_{32})-4g_1g_2u_3{}^*+2\,(z_4D^1-z_0D^2+z_0z_2+z_1z_3)\,u_1{}^*+2\,(z_3D^2-z_0D^1+z_0z_1+z_2z_4)\,u_2{}^*$$

In what follows, we shall consider, instead of the system (2.12) of three second order differential equations, a system of six first order equations in the unknowns $u_k{}^*$ and σ_{3k}, consisting of the equations (2.8) and (2.13).

3. Construction of applied theories. The variables $u_k{}^*$ and dimensionless stresses σ_{sk} are analytic functions of the coordinates of the points belonging to the region V. Consequently, they can be expanded into power series in terms of the coordinate

$$u_k{}^*=\sum_{s=0}^{\infty}\frac{t^s}{s!}\left(\frac{\partial^s u_k{}^*}{\partial t^s}\right)\Bigg|_{t=0},\quad \sigma_{ik}=\sum_{s=0}^{\infty}\frac{t^s}{s!}\left(\frac{\partial^s\sigma_{ik}}{\partial t^s}\right)\Bigg|_{t=0} \tag{3.1}$$

Then, differentiating with respect to t the left and right parts of the equations (2.8),(2.10) and (2.13) and calculating the resulting relations at $t=0$, we obtain an infinite system of recurrent equations for the unknown coefficients appearing in the series (3.1). Solving the system mentioned above with help of a symbolic notation, we can write the unknown coefficients as the outcome of application of certain operators $A_{k,s}^j,\ldots,B_{ik,s}^j$ to the displacements $u_j=u_j{}^*(\alpha,\beta,0)$ and stresses $\sigma_j=\sigma_{j3}(\alpha,\beta,0)$ given at the middle surface.

Thus we arrive at the expansions of the form

$$u_k{}^*=u_k+\sum_{s=1}^{\infty}t^s(A_{k,s}^j u_j+B_{k,s}^j\sigma_j),\quad \sigma_{pq}=\sum_{s=0}^{\infty}t^s(A_{pq,s}^j u_j+B_{pq,s}^j\sigma_j) \tag{3.2}$$

$$\sigma_{3k}=\sigma_k+\sum_{s=1}^{\infty}t^s(A_{3k,s}^j u_j+B_{3k,s}^j\sigma_j)\quad(p,q=1,\ 2) \tag{3.3}$$

Writing out explicitly the first terms of the series (3.2) and (3.3), we obtain

$$\begin{aligned}u_1{}^*&=u_1+t\,(\sigma_1-\partial_1u_3-k_\alpha u_1)+\ldots\\ u_2{}^*&=u_2+t\,(\sigma_2-\partial_2u_3-2mu_1-k_\beta u_2)+\ldots\\ u_3{}^*&=u_3+t\,\{-f_4\sigma_3-c_4[(\partial_1+a)\,u_1+(\partial_2+b)\,u_2-Hu_3]\}+\ldots\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sigma_{11}&=(4\varkappa\partial_1+2c_4a)\,u_1+(2c_4\partial_2+4\varkappa b)\,u_2+(2k_\beta-4\varkappa H)\,u_3+c_4\sigma_3+t\,\{2\,[m\,(c_4b-c_0\partial_2)+2\varkappa a\,(k_\alpha-k_\beta)-2\varkappa AH_\alpha'+\\&Bm_\beta']\,u_1+(c_8\partial_1+c_4a)\,\sigma_1+2[c_4\,(k_\alpha-k_\beta)\,\partial_2+c_4m\,(a-\partial_1)+B\,(k_\beta-2\varkappa H)_\beta']\,u_2+2\,[k_\alpha\,(k_\beta-2\varkappa H)-3m^2-\\&\partial_1{}^2-b\partial_2-c_4\Delta]\,u_3+(c_4\partial_2+c_8b)\,\sigma_2+c_2k_\alpha\sigma_3\}+\ldots\\ \sigma_{22}&=(2c_4\partial_1+4\varkappa a)\,u_1+(4\varkappa\partial_2+2c_4b)\,u_2+(2k_\alpha-4\varkappa H)\,u_3+c_4\sigma_3+t\,\{2\,[c_8m\,(b-\partial_2)+c_4\,(k_\beta-k_\alpha)\,\partial_1+A\,(k_\alpha-\\&2\varkappa H)_\alpha']\,u_1+(c_4\partial_1+c_8a)\,\sigma_1+2\,[c_4\,(k_\beta-k_\alpha)\,b+c_4m\,(a-\partial_1)+B\,(k_\alpha-2\varkappa H)_\beta']\,u_2+(c_8\partial_2+c_4b)\,\sigma_2+2\,[m^2+\\&k_\beta\,(k_\alpha-2\varkappa H)-c_4\Delta-a\partial_1-\partial_2{}^2]\,u_3+c_2k_\beta\sigma_3\}+\ldots\\ \sigma_{12}&=(\partial_2-b)\,u_1+(\partial_1-a)\,u_2-2mu_3+t\,\{[(k_\beta-k_\alpha)\,\partial_2+2m\,(c_2\partial_1+c_8a)-Am_\alpha'-B\,(k_\alpha)_\beta']\,u_1+[(k_\alpha-k_\beta)\,\partial_1+\\&m\,(4\varkappa\partial_2+2c_4b)-Bm_\beta'-A\,(k_\beta)_\alpha']\,u_2+2\,[a\partial_2-\partial_2\partial_1-m\,(2\varkappa H+k_\beta-k_\alpha)]\,u_3+(\partial_2-b)\,\sigma_1+(\partial_1-a)\,\sigma_2+\\&c_2m\sigma_3\}+\ldots\\ \sigma_{31}&=\sigma_1+t\,\{(H+k_\alpha)\,\sigma_1+2m\sigma_2-c_4\partial_1\sigma_3-[\partial_2\,(\partial_2+b)+4\varkappa\partial_1\,(\partial_1+a)+2k_1k_2]\,u_1+[\partial_2\,(\partial_1+a)-4\varkappa\partial_1\,(\partial_2+b)]\,u_2+\\&[(4\varkappa H-2k_\beta)\,\partial_1+2m\partial_2+4\varkappa AH_\alpha']\,u_3\}+t^2\,\{[f_8a\partial_2-\end{aligned} \tag{3.5}$$

$c_8 b\partial_1 - \varkappa\partial_2\partial_1 + \Omega_0]\,\sigma_2 - [f_7\,(\partial_1^2 + a\partial_1) + {}^1/_2\,(\partial_2^2 + b\partial_2) +$
$\Omega_0]\,\sigma_1 - [(c_{31}k_\alpha + f_5 k_\beta)\,d_1 + c_{40}m\partial_2 + \Omega_0]\,\sigma_3 - [2\varkappa\,(H +$
$2k_\alpha)\,\partial_1^2 + (k_\beta + {}^1/_2 H)\,\partial_2^2 + 8\varkappa m\partial_2\partial_1 + \Omega_1]\,u_1 + [c_2 m\partial_1^2 -$
$(c_{71}k_\alpha + {}^1/_2 k_\beta)\,\partial_2\partial_1 - c_{92}m\partial_2^2 + \Omega_1]\,u_2 + [2\varkappa\partial_1\Delta + \Omega_1]\,u_3\} +$
$t^3\,\{[d_5\partial_1\Delta + \Omega_1]\,\sigma_3 - [m\partial_1^2 + (2\varkappa k_\alpha + \partial_8 k_\beta)\,\partial_2\partial_1 +$
$m\,(1 + d_{64})\,\partial_2^2 + \Omega_1]\,\sigma_2 - [(c_9 k_\alpha + d_8 k_\beta)\,\partial_1^2 + m\,(2 +$
$d_{64})\,\partial_2\partial_1 + (k_\beta + {}^1/_2 k_\alpha)\,\partial_2^2 + \Omega_1]\,\sigma_1 + [{}^4/_3\varkappa\partial_1\Delta\,(\partial_1 + a) +$
${}^1/_6\partial_2\Delta\,(\partial_2 + b) + \Omega_2]\,u_1 + [{}^4/_3\varkappa\partial_1\Delta\,(\partial_2 + b) - {}^1/_6\partial_2\Delta\,(\partial_1 +$
$a) + \Omega_2]\,u_2 + [(d_{76}k_\alpha + {}^1/_3 k_\beta)\,\partial_1^3 + 2m\,(d_{67}\partial_2\partial_1^2 + d_{63}\partial_2^3) +$
$(d_{72}k_\alpha + k_\beta)\,\partial_2^2\partial_1 + \Omega_2]\,u_3\} + \ldots$

$\sigma_{32} = \sigma_2 + t\,\{(H + k_\beta)\,\sigma_2 - c_4\partial_2\sigma_3 + [\partial_1\,(\partial_2 + b) -$
$4\varkappa\partial_2\,(\partial_1 + a)]\,u_1 - [\partial_1\,(\partial_1 + a) + 4\varkappa\partial_2\,(\partial_2 + b) + 2k_1k_2]\,u_2 +$
$[(4\varkappa H - 2k_\alpha)\,\partial_2 + 2m\partial_1 + 4\varkappa BH_\beta']\,u_3\} + t^2\,\{[b\partial_1 -$
$f_8 a\partial_2 - \varkappa\partial_2\partial_1 + \Omega_0]\,\sigma_1 - [{}^1/_2\,(\partial_1^2 + a\partial_1) + f_7\,(\partial_2^2 + b\partial_2) +$
$\Omega_0]\,\sigma_2 - [f_4 m\partial_1 + (f_5 k_\alpha + c_{31}k_\beta)\,\partial_2 + \Omega_0]\,\sigma_3 + [m\,(2\varkappa\partial_2^2 -$
$c_2\partial_1^2) - ({}^1/_2 k_\alpha + c_{71}k_\beta)\,\partial_2\partial_1 + \Omega_1]\,u_1 - [(k_\alpha + {}^1/_2 H)\,\partial_1^2 +$
$2m\partial_2\partial_1 + 2\varkappa\,(H + 2k_\beta)\,\partial_2^2 + \Omega_1]\,u_2 + [2\varkappa\partial_2\Delta + \Omega_1]\,u_3\} +$
$t^3\,\{[d_5\partial_2\Delta + \Omega_1]\,\sigma_3 - [(k_\alpha + {}^1/_2 k_\beta)\,\partial_1^2 + m\,(2 + d_4)\,\partial_2\partial_1 +$
$(d_8 k_\alpha + c_9 k_\beta)\,\partial_2^2 + \Omega_1)]\,\sigma_2 - [d_{04}m\partial_1^2 + (2\varkappa k_\beta + d_8 k_\alpha)\,\partial_2\partial_1 +$
$\Omega_1]\,\sigma_1 + [{}^4/_3\varkappa\partial_2\Delta\,(\partial_1 + a) - {}^1/_6\partial_1\Delta\,(\partial_2 + b) + \Omega_2]\,u_1 +$
$[{}^1/_6\partial_1\Delta\,(\partial_1 + a) + {}^4/_3\varkappa\partial_2\Delta\,(\partial_2 + b) + \Omega_2]\,u_2 + [m\,(d_0\partial_1^3 +$
$d_8\partial_2^2\partial_1) + (k_\alpha + d_{72}k_\beta)\,\partial_2\partial_1^2 + ({}^1/_3 k_\alpha + d_{76}k_\beta)\,\partial_2^3 + \Omega_2]\,u_3\} + \cdots$

$\sigma_{33} = \sigma_3 + t\,\{[2k_\beta - 4\varkappa H)\,\partial_1 + 2m\,(b - \partial_2) + a\,(2k_\alpha -$
$4\varkappa H)]\,u_1 + [(2k_\alpha - 4\varkappa H)\,\partial_2 + 2m\,(a - \partial_1) + b\,(2k_\beta -$
$4\varkappa H)]\,u_2 + (4\varkappa H^2 - 4k_1k_2)\,u_3 - (\partial_1 + a)\,\sigma_1 - (\partial_2 +$
$b)\,\sigma_2 - c_2 H\sigma_3\} + t^2\,\{[f_5\Delta + \Omega_0]\,\sigma_3 - [2m\partial_1 + (2k_\beta +$
$f_7 H)\,\partial_2 + \Omega_0]\,\sigma_2 - [(2k_\alpha + f_7 H)\,\partial_1 + 2m\partial_2 + \Omega_0]\,\sigma_1 +$
$[2\varkappa\Delta\,(\partial_1 + a) + \Omega_1]\,u_1 + [2\varkappa\Delta\,(\partial_2 + b) + \Omega_1]\,u_2 -$
$[4\varkappa\,(AH_\alpha'\partial_1 + BH_\beta'\partial_2) + \Omega_0]\,u_3\} + t^3\,\{[d_{07}\Delta\,(\partial_1 + a) +$
$\Omega_1]\,\sigma_1 + [d_{07}\Delta\,(\partial_2 + b) + \Omega_1]\,\sigma_2 + [c_4\,(k_\alpha + {}^1/_2 H)\,\partial_1^2 +$
$c_{32}m\partial_2\partial_1 + c_4(k_\beta + {}^1/_2 H)\,\partial_2^2 + \Omega_1]\,\sigma_3 - [{}^2/_3\varkappa\Delta\Delta + \Omega_2]\,u_3 +$
$[-md_0\partial_1^3 + 2\,(c_{25}k_\alpha + d_8 k_\beta)\,\partial_2\partial_1^2 + m\,(6\varkappa + d_{34})\,\partial_2^2\partial_1 +$
$(d_{54}H + d_{98}k_\beta)\,\partial_2^3 + \Omega_2]\,u_2 + [(d_{54}H + d_{98}\,k_\alpha)\,\partial_1^3 +$
$m\,(6\varkappa + d_{34})\,\partial_2\partial_1^2 + 2\,(d_8 k_\alpha + C_{25}k_\beta)\,\partial_2^2\partial_1 - md_0\partial_2^3 + \Omega_2]\,u_1\} + \cdots$

$(\Delta = \partial_1^2 + a\partial_1 + \partial_2^2 + b\partial_2,\quad (D^1)\,|_{t=0} = A\partial\,/\,\partial\alpha \equiv \partial_1$
$(D^2)\,|_{t=0} = B\partial\,/\,\partial\beta \equiv \partial_2)$

Here Ω_k denote the operators of order lower than those given (k is the order of the operator Ω_k), while a and b are the geodesic curvatures of the coordinate lines and the middle surface. We also use the following set of notations:

$$1\,/\,\sqrt{E} = A,\quad 1\,/\,\sqrt{G} = B,\quad (z_1)\,|_{t=0} = -A\,(\ln B)_\alpha' = a\,,\quad (z_2)\,|_{t=0} = -B\,(\ln A)_\beta' = b$$

$$C_{rs} = (1 + r)\,\varkappa - 3 + s\,/\,2,\quad d_{rs} = c_{rs}\,/\,3,\quad c_{1s} \equiv c_s,\quad d_{1s} \equiv d_s,\quad c_{0s} \equiv f_s$$

In addition to (2.3), we note other relationships which facilitate the derivation of the expansions (3.4) and (3.5). We have

$$(D^1)_t' = z_3 D^1 + 2z_0 D^2,\quad (D^2)_t' = z_4 D^2,\quad (z_2)_t' = z_2 z_4 - D^2 z_3$$

$$(z_0)\,|_{t=0} = m,\quad (z_3)\,|_{t=0} = k_\alpha,\quad (z_4)\,|_{t=0} = k_\beta,\quad (D^1 + z_1)\,D^2 \equiv (D^2 + z_2)\,D^1$$

Further, we introduce into our discussion the specific forces T_p, S_{qp}, N_p and moments G_p, H_{qp} appearing on the coordinate cross-sections $x^p = \mathrm{const}$ $(p \neq q = 1,2)$ of the shell. We have

$$T_p\mathbf{i}_{p0} + S_{qp}\mathbf{i}_{q0} - N_p\mathbf{i}_3 = \Big(\int_{-h}^{h} \boldsymbol{\sigma}_{(p)}\sqrt{g_{qq}}\,dt\Big)\big(\sqrt{g_{qq}}\big)^{-1}\big|_{t=0},\quad \mathbf{i}_{p0} = (\mathbf{i}_p)|_{t=0} \tag{3.6}$$

$$(-1)^q\,(H_{qp}\mathbf{i}_{p0} + G_p\mathbf{i}_{q0}) = \Big(\int_{-h}^{h} (\boldsymbol{\sigma}_{(p)} \times \mathbf{i}_3)\,t\sqrt{g_{qq}}\,dt\Big)\big(\sqrt{g_{qq}}\big)^{-1}\big|_{t=0}$$

$$\boldsymbol{\sigma}_{(1)} = \mathbf{p}_1,\quad \boldsymbol{\sigma}_{(2)} = \big(\sqrt{g}\,\mathbf{p}_2 - g_{12}\mathbf{p}_1\big)\,/\sqrt{g_{11}g_{22}}\,,\quad \mathbf{i}_p = (1 - k_\beta t)\,\mathbf{i}_{p0} + (-1)^q\,mt\mathbf{i}_{q0}]\sqrt{G\,/\,g_{22}}\quad (p \neq q = 1,\,2)$$

Here $\boldsymbol{\sigma}_{(p)}$ is the stress vector at the surface $x^p = \mathrm{const}$.

Taking into account (1.5) and (2.4) we obtain, from (3.6),

$$N_p = -\mu \int_{-h}^{h} \{\sigma_{p3} + t(\delta_p m\sigma_{q3} - k_{qq}\sigma_{p3}) + t^2[{}^1/_2\delta_p m\,(k_{pp} - k_{qq})\,\sigma_{q3} + m^2\sigma_{p3}({}^1/_2 - \delta_p)] + \ldots\}\,dt \tag{3.7}$$

$$T_p = \mu \int_{-h}^{h} L_p\,dt, \quad S_{qp} = \mu \int_{-h}^{h} M_{qp}\,dt, \quad H_{qp} = \mu \int_{-h}^{h} M_{qp} t\,dt, \quad G_p = -\mu \int_{-h}^{h} L_p t\,dt$$

$$M_{qp} = \sigma_{qp}' + t\,[\delta_p m\sigma_{qq} + (-1)^q\, m\sigma_{pp} - k_{qq}\sigma_{qp}] + \ldots, \quad (k_{11} = k_\alpha,\ k_{22} = k_\beta)$$

$$L_p = \sigma_{pp} + t\,[(2\delta_p - 1)\,m\sigma_{qp} - k_{qq}\,\sigma_{pp}] + \ldots, \quad (1 + (-1)^p = \delta_p)$$

Now replacing in (3.7) the stresses σ_{pk} by the corresponding expressions from (3.4) and (3.5) and integrating, we obtain the following expressions in powers of h^2:

$$\begin{aligned}
N_p &= 2\mu h\,[-\sigma_p - {}^1/_3 h^2\,(C_{p,2}^j u_j + D_{p,2}^j\sigma_j) + \ldots] \\
S_{qp} &= 2\mu h\,(\omega + h^2 S_{qp,2} + \ldots) \\
T_p &= 4\mu h\,(c_6\varepsilon_p + C_4\varepsilon_q + f_5\sigma_3 + h^2 T_{p,2} + \ldots) \\
H_{qp} &= {}^4/_3\mu h^3\,[\tau + {}^1/_2 k_{pp}\ \omega + 2m\,(c_6\varepsilon_q + c_5\varepsilon_p) + {}^1/_2\,(\partial_2 - b)\times \\
&\quad \sigma_1 + {}^1/_2\,(\partial_1 - a)\,\sigma_2 + mc_3\sigma_3 + h^2 H_{qp,2} + \ldots] \\
G_p &= -{}^4/_3\mu h^3\,[c_6\varkappa_p + c_4\varkappa_q + (k_{pp} - k_{qq})(c_6\varepsilon_p + c_4\varepsilon_q) + \\
&\quad m\omega + (f_7\partial_p + f_5 k_p^*)\,\sigma_p + (f_5\partial_q + f_7 k_q^*)\,\sigma_q + (f_4 k_{pp} - \\
&\quad f_5 k_{qq})\,\sigma_3 + h^2 G_{p,2} + \ldots] \quad (k_1^* = a,\ k_2^* = b)
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\varepsilon_p &= \partial_p u_p + k_q^* u_q - k_{pp} u_3, \quad \omega = (\partial_2 - b)\,u_1 + (\partial_1 - a)\,u_2 - 2mu_3 \\
\tau &= k_\beta\,(a - \partial_1)\,u_2 + k_\alpha\,(b - \partial_2)\,u_1 - m\,(\varepsilon_1 + \varepsilon_2) - u_1\partial_1 m - \\
&\quad u_2\partial_2 m + (b\partial_1 - \partial_1\partial_2)\,u_3 \\
\varkappa_p &= {}^1/_2 m\,[(k_q^* + \partial q)\,u_p - (k_p^* + \partial_p)\,u_q] - k_q^*\,(\partial_q u_3 + \\
&\quad k_{qq} u_q + mu_p) - \partial_p\,(\partial_p u_3 + k_{pp} u_p + mu_q) \\
C_{p,2}^j &= A_{3p,2}^j - k_{qq} A_{3p,1}^j + \delta_p m A_{3q,1}^j
\end{aligned} \tag{3.9}$$

where $\varepsilon_1, \omega, \varepsilon_2$ and $\varkappa_1, \tau, \varkappa_2$ are the components of the tangential and flexural deformation of the middle surface, respectively.

The expansions (3.2) and (3.8) are of use in constructing a number of applied theories of various purpose and accuracy. Let us denote by Γ_1 and Γ_2 the parts of the surface S^* defined, respectively, by the equations $t = \pm h$ and $\varphi(\alpha, \beta) = 0$ where h is the half-thickness of the shell. Theories which leave the boundary of Γ_1 stress-free and allow the conditions at the boundary of Γ_2 to be satisfied are not dealt with here (see e.g. /5/). Below we present a method of constructing the theories which will ensure that the conditions at the boundary of Γ_1 are satisfied without worrying about the boundary conditions at the boundary of Γ_2.

Let the vector $\mathbf{q}$ defined at the surface Γ_1 have the form

$$\mathbf{q}^+ = \sum_{k=1}^{3} q_k^+\mathbf{i}_k \quad \text{when} \quad t = +h, \quad \mathbf{q}^- = \sum_{k=1}^{3} q_k^-\mathbf{i}_k \quad \text{when} \quad t = -h$$

By virtue of (3.3) we write the boundary conditions on Γ_1 in the form

$$q_k^+/\mu = \sigma_k + \sum_{s=1}^{\infty} h^s\,(A_{3k,\,s}^j u_j + B_{3k,\,s}^j\sigma_j), \quad -q_k^-/\mu = \sigma_k + \sum_{s=1}^{\infty} (-h)^s\,(A_{3k,\,s}^j u_j + B_{3k,\,s}^j\sigma_j)$$

and this yields the equivalent system

$$\begin{aligned}
Q_k &= (q_k^+ - q_k^-)/2\mu = \sigma_k + h^2\,(A_{3k,\,2}^j u_j + B_{3k,\,2}^j\sigma_j) + \ldots \\
P_k &= (q_k^+ + q_k^-)/2\mu = h\,(A_{3k,\,1}^j u_j + B_{3k,\,1}^j\sigma_j) + h^3\,(A_{3k,\,3}^j u_j + B_{3k,\,3}^j\sigma_j) + \ldots
\end{aligned} \tag{3.10}$$

In the case of small h, we shall seek the unknown u_j and σ_j in the form of the asymptotic expansions

$$u_j = \sum_{s=0} h^{s-1} u_j^{(s)}, \quad \sigma_j = \sum_{s=0} h^s\sigma_j^{(s)} \tag{3.11}$$

Substituting the expansions (3.11) into the equations (3.10) and equating the terms accompanying $h^s\,(s = 0, 1, 2, \ldots)$, we obtain a system of recurrence relations for the functions $u_j^{(s)}$ and $\sigma_j^{(s)}$. The relations arising at $s = 0$ and 1 define an applied theory with an error of the order of h^2 as compared with unity. We have

$$A^j_{3k,1}u^{(0)}_j = P_k, \quad A^j_{3k,1}u^{(1)}_j = -B^j_{3k,1}Q_j, \quad \sigma^{(0)}_k = Q_k, \quad \sigma^{(1)}_k = -A^j_{3k,2}u^{(0)}_j \quad (k=1,2,3) \tag{3.12}$$

Equations (3.12) can be written in a different form by taking into account the relations

$$A^j_{3p,1}u_j = 2k_p^*(\varepsilon_q - \varepsilon_p) - 2\partial_p(c_6\varepsilon_p + c_4\varepsilon_q) - (\partial_q + 2k_q^*)\omega$$

$$A^j_{33,1}u_j = 2(k_\alpha\varepsilon_2 + k_\beta\varepsilon_1) - 2c_6H(\varepsilon_1+\varepsilon_2) - 2m\omega \ (p \neq q = 1,2)$$

Now, substituting (3.11) into the expansions (3.8) and taking into account (3.12), we obtain

$$\begin{aligned} T_p &= 4\mu\,[c_6\varepsilon_p^{(0)} + c_4\varepsilon_q^{(0)} + h(c_6\varepsilon_p^{(1)} + c_4\varepsilon_q^{(1)} + f_5Q_3) + \ldots] \\ S_{qp} &= 2\mu\,(\omega^{(0)} + h\omega^{(1)} + \ldots) \\ N_p &= 2\mu h\,[-Q_p - h(^2/_3\sigma_p^{(1)} - {}^1/_3k_{qq}P_p + {}^1/_3\delta_p mP_q) + \ldots] \\ H_{qp} &= {}^4/_3\mu h^2\,\{\tau^{(1)} + {}^1/_2k_{pp}\omega^{(0)} + 2m(c_6\varepsilon_q^{(0)} + c_5\varepsilon_p^{(0)}) + \\ &\quad h[\tau^{(1)} + {}^1/_2k_{pp}\omega^{(1)} + 2m(c_6\varepsilon_q^{(1)} + c_5\varepsilon_p^{(1)}) + {}^1/_2(\partial_2 - b)Q_1 + \\ &\quad {}^1/_2(\partial_1 - a)Q_2 + mc_3Q_3] + \ldots\} \\ G_p &= -{}^4/_3\mu h^2\,\{c_6\varkappa_p^{(0)} + c_4\varkappa_q^{(0)} + (k_{pp} - k_{qq})(c_6\varepsilon_p^{(0)} + c_4\varepsilon_q^{(0)}) + \\ &\quad m\omega^{(0)} + h[c_6\varkappa_p^{(1)} + c_4\varkappa_q^{(1)} + m\omega^{(1)} + (k_{pp} - k_{qq})(c_6\varepsilon_p^{(1)} + \\ &\quad c_4\varepsilon^1_q) + (f_7\partial_p + f_5k_p^*)Q_p + (f_5\partial_q + f_7k_q^*)Q_q + (f_1k_{pp} - f_5k_{qq})Q_3] + \ldots\} \end{aligned} \tag{3.13}$$

where the quantities $\varepsilon_p^{(s)}, \ldots, \varkappa_p^{(s)}$ are determined in terms of the functions $u_j^{(s)}$ according to the formulas (3.9). In addition we have the expansions

$$\begin{aligned} u_p^0 &= h^{-1}u_p^{(0)} + u_p^{(1)} - \zeta(\partial_p u_3^{(1)} + k_{pp}u_p^{(0)} + mu_q^{(0)}) + \ldots \\ u_3^0 &= h^{-1}u_3^{(0)} + u_3^{(1)} - \zeta c_4(\varepsilon_1^{(0)} + \varepsilon_2^{(0)}) + \ldots \\ &(\zeta = t/h, \ \mathbf{U} = u_1^0\mathbf{i}_{10} + u_2^0\mathbf{i}_{20} + u_3^0\mathbf{i}_3) \end{aligned} \tag{3.14}$$

Relations (3.9), (3.12) and asymptotic expansions (3.13), (3.14) with an error of the order of h^2 compared with unity, together form a complete refined system of "two-dimensional" equations of the theory of shells. The applied theory constructed enables us to form an opinion regarding the accuracy of the theories of thin shells used. In particular, we shall find how the equations (3.12)—(3.14) agree with the equations of the Gol'denveizer iterative theory of shells, which can be written using the notation adopted in this paper, in the form

$$\begin{aligned} &(2\mu)^{-1}(k_{qq}N_p + mN_q) - hA^j_{3p,1}u_j = -P_p + h(HQ_p - c_4\partial_pQ_3) + \ldots \\ &(2\mu)^{-1}[(\partial_1 + a)N_1 + (\partial_2 + b)N_2] + hA^j_{33,1}u_j = P_3 + hc_2HQ_3 + \ldots \\ &(2\mu)^{-1}N_p + h^3F_p{}^ju_j = -hQ_p + h^2(HP_p - d_4\partial_pP_3) + \ldots \quad (p \neq q = 1,2) \end{aligned} \tag{3.15}$$

where $F_p{}^j$ are known operators. The relations (3.15) can be arrived at when the forces T_p, S_{qp} and moments G_p, H_{qp} defined in the monograph /6/ for the case of tri-orthogonal coordinate system by the equations

$$T_p = 4\mu h(c_6\varepsilon_p + c_4\varepsilon_q + f_5Q_3), \ S_{qp} = 2\mu h\omega, \ H_{qp} = {}^4/_3\mu h^3(\tau + {}^1/_2k_p\omega) \tag{3.16}$$

$$G_p = -{}^4/_3\mu h^3[c_6\varkappa_p + c_4\varkappa_q + c_6(k_p - k_q)\varepsilon_p + c_4(c_6H - k_q)(\varepsilon_1+\varepsilon_2) + h^{-1}f_5P_3], \ (p \neq q = 1,2, \ k_{pp} = k_p, \ m = 0)$$

are expressed, in the equations of equilibrium, in terms of the displacements u_j of the middle surface, and the terms containing the intersecting conditions N_p are retained.

Seeking the unknown u_j and N_p in the form of expansions

$$u_j = \sum_{s=0} h^{s-1}v_j^{(s)}, \quad N_p = \sum_{s=0} h^{s+1}N_p^{(s)} \tag{3.17}$$

and substituting (3.17) into (3.15), we obtain a system of equations which yield, at small h, the relations

$$A^j_{3k,1}v^{(0)}_j = P_k, \ A^j_{3k,1}v^{(1)}_j = -B^j_{3k,1}Q_j \ (k=1,2,3), \quad N_p = 2\mu h[-Q_p - h(F_p{}^jv_j^{(0)} - HP_p + d_4\partial_pP_3) + \ldots] \tag{3.18}$$

If the functions $u_j^{(s)}$ and $v_j^{(s)}$ are subjected to the same boundary conditions, then, equating (3.12) and (3.13) with (3.18) we can establish that $u_j^{(s)} = v_j^{(s)} \ (s = 0, 1)$, while the only coincident terms in the intersecting forces N_p are the principal terms of the corresponding asymptotic expansions.

Further, substituting (3.17) into (3.16) and taking into account

$$u_j^{(s)} = v_j^{(s)} \ (s = 0,1), \ f_5P_3 = c_4(k_p\varepsilon_q^{(0)} + k_q\varepsilon_p^{(0)}) - c_4c_6H(\varepsilon_p^{(0)} + \varepsilon_q^{(0)})$$

we obtain

$$\begin{aligned} H_{qp} &= {}^4/_3\mu h^2[\tau^{(0)} + {}^1/_2k_p\omega^{(0)} + h(\tau^{(1)} + {}^1/_2k_p\omega^{(1)}) + \ldots] \\ T_p &= 4\mu[c_6\varepsilon_p^{(0)} + c_4\varepsilon_q^{(0)} + h(c_6\varepsilon_p^{(1)} + c_4\varepsilon_q^{(1)} + f_5Q_3) + \ldots], \quad S_{qp} = 2\mu(\omega^{(0)} + h\omega^{(1)} + \ldots) \\ G_p &= -{}^4/_3\mu h^2\{c_6\varkappa_p^{(0)} + c_4\varkappa_q^{(0)} + (k_p - k_q)(c_6\varepsilon_p^{(0)} + c_4\varepsilon_q^{(0)}) + \\ &\quad h[c_6\varkappa_p^{(1)} + c_4\varkappa_q^{(1)} + c_6(k_p - k_q)\varepsilon_p^{(1)} + c_4(c_6H - k_q)(\varepsilon_p^{(1)} + \varepsilon_q^{(1)})] + \ldots\}, \quad (p \neq q = 1,2) \end{aligned} \tag{3.19}$$

Equating (3.19) with (3.13) we find, that for the case $m = 0$, $k_{pp} = k_p$, the forces T_p and S_{qp} determined according to various theories, show the same error of the order of h^2 compared with unity. As regards the moments G_p and H_{qp}, here we observe that only the principal terms of the corresponding expansions coincide. Finally, comparing the displacements u_j^0 described by the iterative theory with the expansions (3.14), we can establish that they both are of the same accuracy.

Thus, comparing the theory constructed here with the Gol'denveizer's iterative theory, we find that the accuracy of determination of the quantities N_p, G_p and H_{qp} is improved by one order of magnitude.

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